A GENERAL THEORY OF LINEAR TIME INVARIANT ADAPTIVE FEEDFORWARD SYSTEMS WITH HARMONIC REGRESSORS

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Abstract

This paper establishes necessary and sufficient conditions for an adaptive system with a harmonic regressor (i.e., a regressor comprised exclusively of sinusoidal signals) to admit an exact linear time-invariant (LTI) representation. These conditions are important because a large number of adaptive systems used in practice have sinusoidal regressors, and the stability, convergence and robustness properties of systems having LTI representations can be completely analyzed by well-known methods. The theory is extended to applications where the LTI conditions do not hold, in which case the harmonic adaptive system can be written as the parallel connection of a purely linear time-invariant (LTI) subsystem and a linear time-varying (LTV) subsystem, An explicit upper bound is established on the induced 2-norm of the LTV block which allows systematic treatment using robust control methods applicable to LTI systems with norm-bounded additive perturbations.

1 INTRODUCTION

A large number of adaptive systems used in practice (e.g., for adaptive signal processing, noise canceling, acoustics, vibration suppression, etc.), have regressors which cent ain sinusoidal excitations. In certain interesting cases, such systems have been found to admit exact - finite-dimensional linear time-invariant (LTI) representations (cf., Glover [14], Morgan and Sanford [18], Morgan [19], Elliott *et. al. [II],* and Widrow and Stearns [25], Bodson *et. al.* [8], Messner and Bodson [16]). Such cases are important because the stability, convergence and robustness properties of the adaptive system can be analyzed simply and completely within an LTI framework.

Interestingly, despite various successes in specific application areas, no general unified theory of LTI adaptive feedforward systems has emerged. In particular, no definitive conditions for the LTI phenomena have been previously established.

In this paper, a general unified theory of LTI representations is developed for adaptive systems having harmonic regressors. The main result is a precise condition (i.e., both necessary and sufficient), for such harmonic adaptive systems to have an exact LTI representation, and a closed-form analytic expression for this LTI representation when the condition is satisfied. The theory completely unifies existing results by reproducing as special cases all known instances of LTI adaptive systems found in the literature. More importantly, the theory is formulated in a very general framework and indicates a much larger class of LTI adaptive systems than previously known.

The theory is then extended to deal with harmonic adaptive systems for which the LTI conditions are not satisfied. In such cases, it is shown that the adaptive system can be written as the parallel connection of a purely LTI block and an LTV block. An explicit upper bound is determined on the *2-norm* of the LTV block. It is demonstrated that the upper bound can be maximally tightened by solving a related linear matrix inequality (LMI). The resulting adaptive system has the form of an LTI plant with a norm-bounded additive perturbation which can be analyzed using standard robust control formulations.

The LTI approach for analyzing adaptive systems represents a strong departure from traditional approaches based primarily on nonlinear and/or time-varying representations and stability theory [21]. In contrast, the LTI approach provides a more complete characterization of the overall system behavior, which can potentially lead to the design of adaptive systems with better performance and robustness properties.

All results in this paper are based on the analysis in a recent report [2] and related conference papers [3][4][5].

2 BACKGROUND

2.1 Adapt ive Systems with Harmonic Regressors

The configuration to be studied is shown in Figure 2.1, An estimate $\hat{y}$ of some signal y is to be constructed as a linear combination of the elements of a regressor vector $x(t) \in R^N$, i.e.,

Estimated Signal

$$\hat{y} = w(t)^T x(t) \tag{2.1}$$

where $w(t) \in R^N$ is a parameter vector which is tuned in real-time using the adaptation algorithm,

Adaptation Algorithm

$$w = \mu\Gamma(p)[\tilde{x}(t)e(t)] \tag{2.2}$$

Here, the notation $\Gamma(p)[\cdot]$ is used to denote the multivariable LTI transfer function $\Gamma(s)$ .1 where $\Gamma(s)$ is any LTI transfer function in the Laplace s operator (the differential operator p will replace the Laplace operator s in all time-domain filtering expressions); the term $e(t) \in R^1$ is an error signal; $\mu > 0$ is an adaptation gain; and the signal $\tilde{x}$ is obtained by filtering the regressor x through any stable filter $F(p)$, i.e., *Regressor Filtering*

$$\tilde{x} = F(p)[x] \tag{2.3}$$

The notation $F(p)[\cdot]$ denotes the multivariable LTI transfer function $F(s) \cdot I$ with S1S0 filter $F(s)$, acting on the indicated vector time domain signal.

For the purposes of this paper, it will be assumed that the regressor x can be written as a linear combination of m distinct sinusoidal components $\{\omega_i\}_{i=1}^{m}$, $O < \omega_1 < \omega_2 < \ldots < \omega_m$, where the frequencies have been ordered by size from smallest to largest. Equivalently, it is assumed that there exists a matrix $\mathcal{X} \in R^{Nx2m}$ such that,

Harmonic *Regressor*

$$x = xc(t) \tag{2,4}$$

$$c(t) = \text{[sin(uIt),}\ \cos(\omega_1 t), \ldots \sin(\omega_m t),\ \cos(\omega_m t)]^T \in R^{2m} \tag{2.5}$$

DEFINITION 2.1 *The matrix $\mathcal{X}^T\mathcal{X}$ is defined as the* confluence matrix associated *with the harmonic regressor x in (2.4).* ■

The name "confluence matrix" has been chosen to reflect the fact that for overpararnetrized regressors $x \in RN$, $N > 2m$, N signal channels are effectively combined into a smaller number of 2m channels using properties of this matrix. The confluence matrix has been shown in [6] to play a critical role in determining exponential convergence properties of overparametrized adaptive systems, and will be shown here to play an equally critical role in determining their LTI properties.

Equations (2.1)-(2.5) taken together will be referred to as a *harmonic adaptive system*. Collectively, these equations define an important open-loop mapping from the error signal e to the estimated output $\hat{y}$. Because of its importance, this mapping will be denoted by the special character 7-t, i.e.,

$$\hat{y} = \mathcal{H}[e] \tag{2.6}$$

The special structure of $\mathcal{H}$ is depicted in Figure 2.1.

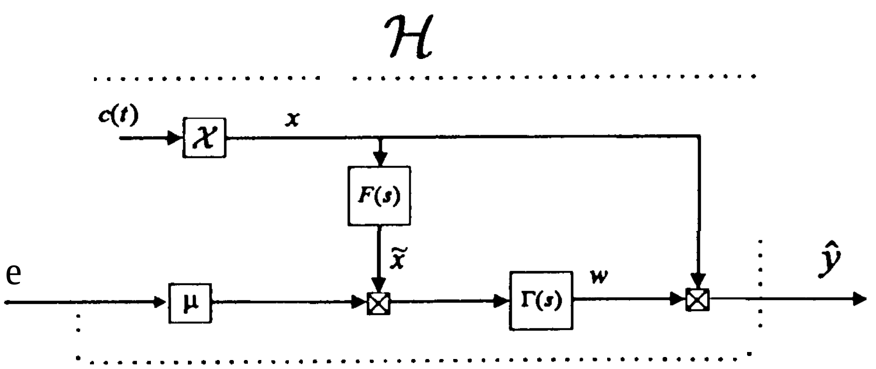


Figure 2.1: LTV operator $\hat{y} = \mathcal{H}[e]$ for adaptive system with harmonic regressor x, adaptation law I'(s), and regressor filter $F(s)$

Most generally $\mathcal{H}$ is a linear time-varying (LTV) operator. However, the main results of this paper show that under certain simple conditions on the matrix X, the mapping $\mathcal{H}$ is actually linear time-invariant (LTI). This result has profound implications for many classes of adaptive systems, since they can be designed and analyzed completely using LTI theory.

REMARK 2.1 The definition of $\Gamma(s)$ is left intentionally general to include analysis of the gradient algorithm (i.e., with the choice $\Gamma(s) = 1/s$), the gradient algorithm with leakage (i.e., $\Gamma(s) = 1/(s+\sigma)$; $\sigma \geq$ O), proportional-plus-integral adaptation (i.e., $\Gamma(s) = k_p+k_i/s$), or arbitrary linear adaptation algorithms of the designer's choosing. Adaptation laws which are nonlinear or normalized (e.g., divided by the norm of the regressor), are not considered here since they do not have an equivalent LTI representation I'(s). ■

REMARK 2.2 The use of the regressor filter $F(s)$ is (2.3) allows the unified treatment of many important adaptation algorithms including the well-known Filtered-X (FX) algorithm from the signal processing literature [25], and the Augmented Error (AE) algorithm of Monopoli [17]. Since x is comprised purely of sinusoidal components and F in (2.3) is stable, all subsequent analysis will assume that the filter output $\tilde{x}$ has reached a steady-state condition. ■

2.2 Discussion

Most generally $\mathcal{H}$ in (2.6) is a *linear* time-varying (LTV) operator, However, under certain conditions on the matrix $\mathcal{X}$, the mapping $\mathcal{H}$ is actually *linear time-invariant.*

The intuition behind this seemingly strange phenomena is explained by the modulation/demodulation properties of multiplicative sinusoidal terms. As a simple example, consider the LTI bandpass filter (BPF) implementation shown in Figure 2.2.

Here, a lowpass filter L(s) is sandwiched between matched sine/cosine multiplications. By inspection, the output can be written in terms of convolution integrals as,

$$Y = \sin\omega_b t \int_0^t \ell(\tau)\sin\omega_b(t-\tau)u(t-\tau)d\tau + \cos\omega_b t \int_0^t \ell(\tau)\cos\omega_b(t-\tau)u(t-\tau)d\tau \quad (2.7)$$

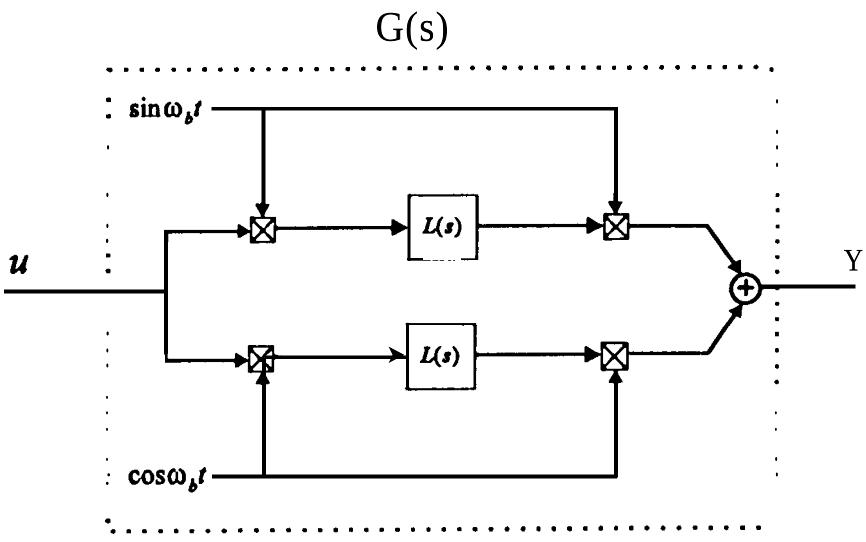
where $\ell(t)$ is the impulse response of the low-pass filter L(s). At first glance this looks like an LTV system, However, substituting the trigonometric identity,

$$\sin\omega_b t \sin\omega_b(t-\tau) + \cos\omega_b t\cos\omega_b(t-\tau) = \cos\omega_b\tau \quad (2.8)$$

into (2.7) and rearranging gives,

$$Y = \int_0^t \ell(\tau)\cos\omega_b(\tau)u(t-\tau)d\tau \quad (2.9)$$

This integral can be recognized as a convolution of the input u with the *time-invariant* impulse response $\ell(t)\cos\omega_b(t)$. Hence, the overall filter is LTI even though it has time-varying elements. The essential relation is identity (2.8) which indicates that the function



$$G(s) = \frac{1}{2}(L(s - j\omega_b) + L(s + j\omega_b))$$

Figure 2.2: Exact LTI Bandpass filter $y = G(p)u$ using Lowpass filter L(s) and modulation properties of sandwiched sinusoidal multiplications

of both t and τ on the left hand side, can be written purely as the function of τ seen on the right hand side.

It is also worth noting that the impulse response of the convolution integral (2.9) is formed by modulating the lowpass filter response f?(t) by $\cos(\omega_b t)$, so that the resulting LTI filter has the bandpass characteristic,

$$G(s) = \mathcal{L}\{\ell(t)\cos\,\omega_b t\} = \frac{1}{2}(L(s - j\omega_b) + L(s + j\omega_b)) \tag{2.10}$$

Here we have used the well-known modulation property $\mathcal{L}\{\ell(t)e^{j\omega_b t}\} = L(s - j\omega_b)$ of the Laplace transform [7].

As a specific example, let L(s) = $1/(s$ + a) in Figure 2.2. Then the operator from u to y shown in Figure 2.2 is exactly representable as an LTI filter, and has a (bandpass) transfer function which can be computed from (2.10) as,

$$G(s) = \frac{s + a}{(s + a)^2 + \omega_b^2} \tag{2.11}$$

3 LTI REPRESENTATIONS

3.1 Single-Tone Regressor Case

Lemma 3.1 first characterizes the LTI operator for the case of a single tone regressor. The arrangement is depicted in Figure 3.1 and corresponds to the special case of $\mathcal{X} = d_i \cdot I_{2\times 2}$ in (2.4).

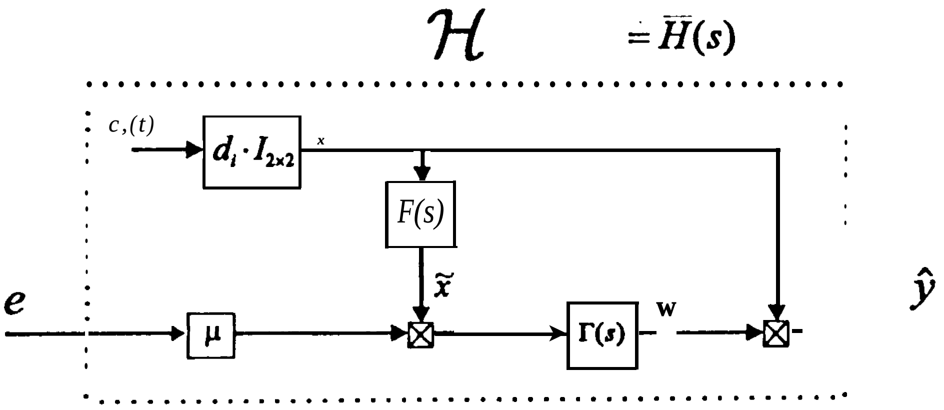


Figure 3.1: Equivalent LTI representation of a harmonic adaptive system with a single tone

LEMMA 3.1 (Single-Tone Regressor) *Let the regressor $x(t)$ in the harmonic adaptive system (2.1)-(2.5) be given by the single-tone expression,*

$$x(t) = d_i c_i(t) \tag{3.1}$$

where d_i is any scalar, and

$$c_i(t) = [\sin \omega_i t, \cos \omega_i t]^T \in R^2 \tag{3.2}$$

Then the mapping $\mathcal{H}$ from e to $\hat{y}$ is exactly representable as the linear time-invariant operator,

$$\mathcal{H} : \hat{y} = \overline{H}(p)e \tag{3.3}$$

where,

$$\overline{H}(s) = \mu d_i^2 \cdot H_i(s) \tag{3.4}$$

$$H_i(s) = \frac{F_R(i)}{2}\Big(\Gamma(s - j\omega_i) + \Gamma(s + j\omega_i)\Big) + \frac{F_I(i)}{2j}\Big(\Gamma(s - j\omega_i) - \Gamma(s + j\omega_i)\Big) \tag{3.5}$$

$$F_R(i) \triangleq \mathbf{Re}(F(j\omega_i));\ F_I(i) \triangleq \mathbf{Im}(F(j\omega_i)) \tag{3.6}$$

Proof:

The filtered regressor (2.3) is composed of a single sinusoid at ω_i put through a linear filter F'(s). Hence, using (3.6) it can be written (in steady-state) as,

$$\tilde{x}(t) = d_i F(p) c_i(t) = d_i \mathcal{F}_i \; c_i(t) \tag{3.7}$$

where,

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2} \tag{3.8}$$

Using (3.7), the mapping from e to $\hat{y}$ can be written as,

$$\hat{y} = \mu d_i^{\,2} c_i(t)^T \text{ « } \Gamma(p) \Big[\mathcal{F}_i c_i(t) e\Big] \tag{3.9}$$

Let $\gamma(t)$ be the impulse response of the LTI operator corresponding to I_'(p). Then using (3.2) and (3.8), equation (3.9) can be expressed in terms of convolution integrals,

$$\begin{aligned} \hat{y} &= \mu d_i^{\,2} \sin\omega_i t \int_0^t \gamma(\tau)\Big[F_R(i)\sin\omega_i(t-\tau) + F_I(i)\cos\omega_i(t-\tau)\Big] e(t-\tau)d\tau \\ &+ \mu d_i^{\,2} \cos\omega_i t \int_0^t \gamma(\tau)\Big[F_R(i)\cos\omega_i(t-\tau) - F_I(i)\sin\omega_i(t-\tau)\Big] e(t-\tau)d\tau \quad (3.10) \\ &= \mu d_i^{\,2} \int_0^t \gamma(\tau)\Big[F_R(i)\cos\omega_i\tau + F_I(i)\sin\omega_i\tau\Big] e(t-\tau)d\tau \quad (3.11) \end{aligned}$$

Here, (3.11) follows from (3.10) by using standard trigonometric identities (see Remark 3.1 below). Note that (3. 11) is in the form of a convolution of the input e(t) with the *time-* invariant impulse response,

$$\bar{\gamma}(t) = \gamma(t)\Big[F_R(i)\cos\omega_i t + F_I(i)\sin\omega_i t\Big] \tag{3.12}$$

Taking the Laplace transform $\mathcal{L}\{\cdot\}$ of (3.12) and using the modulation property [7],

$$\mathcal{L}\{\gamma(t)e^{j\omega_i t}\} = \Gamma(s - j\omega_i) \tag{3.13}$$

gives the desired expression (3.5). ∎

REMARK 3.1 In the proof of Lemma 3.1, (3.11) follows from (3.10) by using trigonometric identity,

$$\begin{aligned} &\sin\omega_i t\Big[F_R(i)\sin\omega_i(t-\tau) + F_I(i)\cos\omega_i(t-\tau)\Big] \\ &+ \cos\omega_i t\Big[F_R(i)\cos\omega_i(t-\tau) - F_I(i)\sin\omega_i(t-\tau)\Big] \\ &= F_R(i)\cos\omega_i\tau + F_I(i)\sin\omega_i\tau \quad (3.14) \end{aligned}$$

Identity (3.14) is a slight generalization of (2.8), and shows that the function of both t and τ on the left-hand side can be represented purely as the function of τ on the right side. This ensures that the convolution (3.1 1) has a shift-invariant kernel, which indicates that the operator from e to $\hat{y}$ is LTI. ■

3.2 Multitone Regressor Case

The main result of this paper is given next which gives necessary and sufficient conditions for the operator $\mathcal{H}$ to be LTI in the general multitone case.

THEOREM 3.1 (LTI Representation Theorem) *Let the regressor $x(t)$ in the harmonic adaptive system (I?. 1)-(%3) be given by the general multitone harmonic expression (2.4)(2.5) where the frequencies $0 < \omega_1 < \ldots < \omega_m$ are distinct, nonzero, and $|F(j\omega_i)| > 0$ for all i.*

Then,

(i) *The mapping $\mathcal{H}$ from e to $\hat{y}$ is exactly representable as the linear time-invariant operator,*

$$\mathcal{H}: \quad \hat{y} = \overline{H}(p)e \tag{3.15}$$

if *and only if the matrix $\mathcal{X}$ in (2.4) satisfies the following,*

$\mathcal{X}$-Orthogonality (XO) Condition:

$$\mathcal{X}^T\mathcal{X} = D^2 \tag{3.16}$$

$$D^2 \triangleq \begin{bmatrix} d_1^2 \cdot I_{2\times 2} & O & \cdots & \hat{0} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_m^2 \cdot I_{2\times 2} \end{bmatrix} \in R^{2m\times 2m} \tag{3.17}$$

where, $d_i^2 \geq 0, i = 1, \ldots m$ are scalars and $I_{2X2} \in R^{2X2}$ is the matrix identity.

(ii) $\overline{H}(s)$ in ($.15) is given in closed-form as,

$$\overline{H}(s) = \mu \sum_{i=1}^{m} d_i^2 \cdot H_i(s) \tag{3.18}$$

$$Hi(s) = \frac{F_R(i)}{2}\Big(\Gamma(s - j\omega_i) + \Gamma(s + j\omega_i)\Big) + \frac{F_I(i)}{2j}\Big(\Gamma(s - j\omega_i) - \Gamma(s + j\omega_i)\Big) \tag{3.19}$$

$$F_R(i) \triangleq \mathrm{Re}(F(j\omega_i)); \quad F_I(i) \triangleq \mathrm{Im}(F(j\omega_i)) \tag{3.20}$$

Proof: See Appendix A. ■

Intuitively, the results of Theorem 3.1 can be understood using the sequence of block diagram rearrangements shown in Figure 3.2, (which incidentally can be taken as an alternative proof of sufficiency, but not necessity). Specifically, Figure 3.2 Part a. shows the initial adaptive system with harmonic regressor; Part b. shows the matrix $\mathcal{X}$ pushed through several scalar matrix blocks of the diagram; Part c. substitutes the identity $\mathcal{X}^T\mathcal{X} = D^*$ where D^* has the special pairwise diagonal form associated with the XO condition (3.16)(3.17); Part d. pushes the matrix D^2 back through several scalar matrix blocks; and Part e. follows by recognizing that Part d. is simply a parallel bank of filters of the form shown in Figure 3.1 each with a perfect sine/cosine basis, i.e., it is representable as a summation of LTI systems of the form treated in Lemma 3.1.

REMARK 3.2 The LTI representation from e to $\hat{y}$ in Theorem 3.1 is invariant under any orthogonal transformation of the regressor, i.e., any $z = Qx$ where $QQ^T = Q^TQ = I$. To see this, assume that $\mathcal{X}^T\mathcal{X} = D^2$, and denote $\mathcal{X}_z = Q\mathcal{X}$. Then using regressor z in the transformed system gives,

$$\mathcal{X}_z^T\mathcal{X}_z = \mathcal{X}^TQ^TQ\mathcal{X} = X^TX = D^2 \tag{3.21}$$

which satisfies the XO condition with the same confluence matrix D^* as the original system.

This invariance is important in light of recent algorithms which perform adaptive filtering in the transform domain, making use of orthogonal regressor transformations of the form $z = Qx$ (cf., the discrete Fourier transform [20] or wavelet transform [12] adaptive filtering approaches). ■

REMARK 3.3 A harmonic adaptive system which does not satisfy the XO condition for a specific X can be made LTI (assuming that $\mathcal{X}^T\mathcal{X}$ is invertible) by the regressor transformation $z = Rx$ where $R = D(\mathcal{X}^T\mathcal{X})^{-1}XT$, and where D is any matrix chosen such that D^2 has the pairwise diagonal form (3.17). Then testing the XO condition for the transformed regressor z gives,

$$\mathcal{X}_z^T\mathcal{X}_z = \mathcal{X}^TR^TR\mathcal{X} \cdot D^2 \tag{3.22}$$

which is satisfied by construction with confluence matrix D*. ■

3.3 Tonal Canonical Form

The proof of Theorem 3.1 (in particular, Part d. of Figure 3.2), indicates a new canonical form.

DEFINITION 3.1 Tonal Canonical Form *is defined as the unique* minimal realization *of an LTI harmonic adaptive system (2. 1)-(1.5) specified by the regressor* choice $\eta = De(t)$ *where* $D \in R^{2m\times 2m}$ *is the non-negative* diagonal *square-root of its confluence matrix* $X^TX = D^2$. ■

Simply stated, any harmonic adaptive *system which admits an LTI representation is* equivalent *to an adaptive system realized in Tonal Canonical Form.* Tonal canonical form

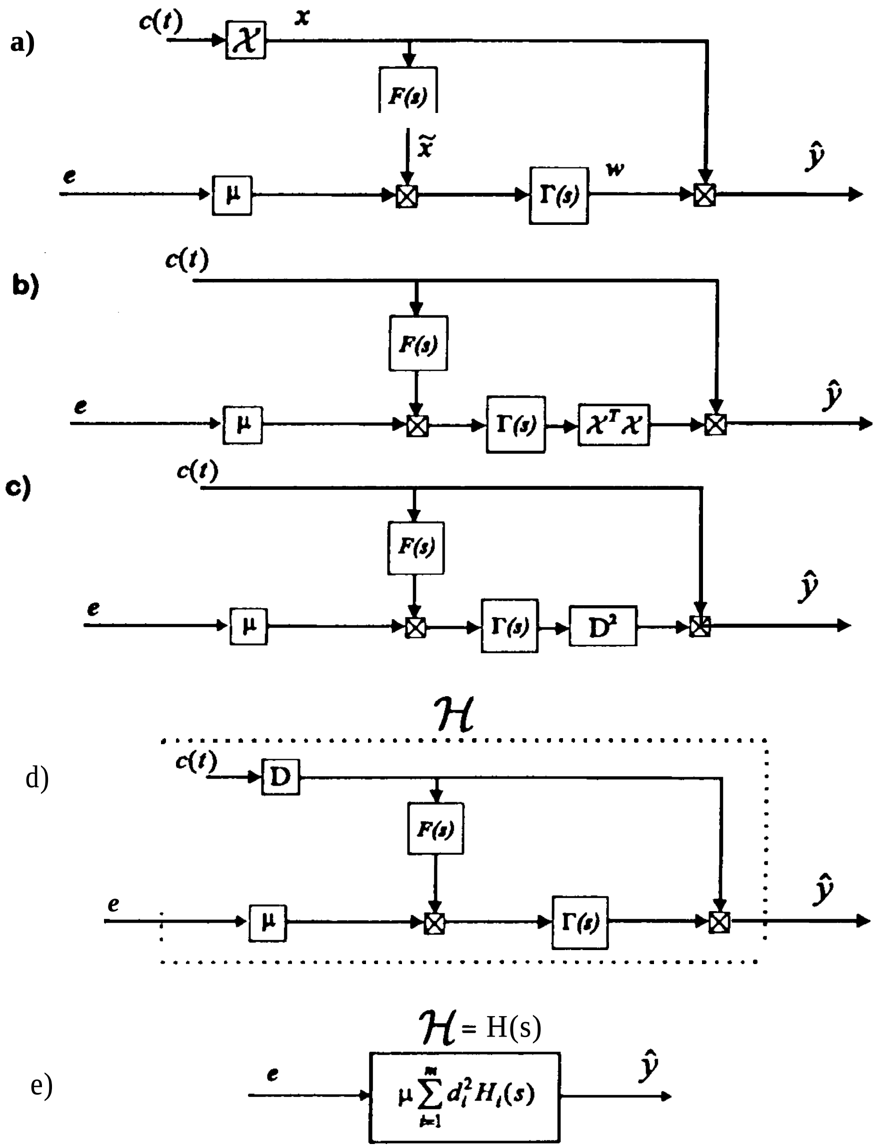


Figure 3.2: The XO condition of Theorem 3.1 motivated by sequence of Block diagram rearrangements

is a canonical form in the sense that it has a minimal *length regressor* (hence fewest possible number of parameters), it *always exists* (by Theorem 3,1 for LTI adaptive systems), and it *is unique* (as *a* consequence of the ordering of the frequencies ω_i in c from low to high),

For convenience, the detailed structure of the Tonal Canonical Form is shown in Figure 3.3 (i.e., obtained by expanding Part d. of Figure 3.2 into scalar terms), The fine structure of Figure 3.3 clearly reveals a parallel bank of second-order LTI systems.

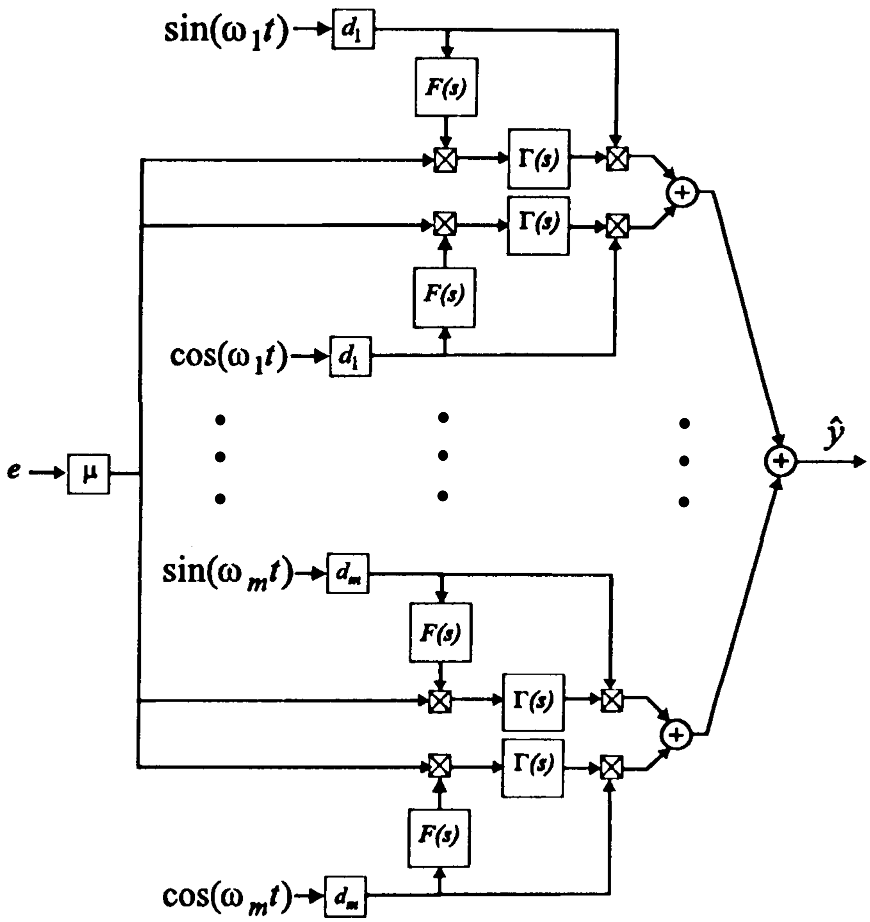


Figure 3,3: Tonal Canonical Form

4 SPECIAL CASES

Several useful LTI representations fall out as special cases of Theorem 3.1, and will be treated in the next few Corollaries.

COROLLARY 4.1 (Gradient Algorithm with Leakage) *Assume that the* adaptive *system with harmonic regressor (1?. 1)-(2.5) is specified as the gradient adaptive algorithm* with leakage, i.e.,

$$\dot{w} = -\sigma w + x(t)e(t) \tag{4.1}$$

for some value of the leakage parameter $\sigma \geq 0$ (cf., Ioannou and Kokotovic [15]). Then, if the XO condition of Theorem S. 1 is satisfied, the LTI expression (S. 18) for $\overline{H}$ is given by,

$$\overline{H}(s) = \mu \sum_{i=1}^{m} d_i{}^2 \cdot \frac{s+\sigma}{s^2 + 2\sigma s + (\omega_i^2 + \sigma^2)} \tag{4.2}$$

Proof: Result (4.2) follows by substituting, $\Gamma(s) = \frac{1}{s+\sigma}$; $\sigma \geq 0$, and $F(s) = 1$ in Theorem 3.1, and rearranging. ∎

COROLLARY 4.2 (Gradient Algorithm) *Assume that the* adaptive *system with harmonic regressor (1?. 1)-(2. 5) is* specified as *the* gradient *adaptive algorithm, i.e.,*

$$\dot{w} = \mu x(t)e(t) \tag{4.3}$$

Then, if the XO condition *of Theorem S.1 is satisfied, the LTI expression* (9.18) *for* $\overline{H}$ *is* given *by,*

$$\overline{H}(s) = \mu \sum_{i=1}^{m} \frac{d_i{}^2 s}{s^2 + \omega_i^2} \tag{4.4}$$

Proof: Result (4.4) follows by substituting $\sigma = 0$ into (4.2) of Corollary 4.1, and rearranging. ∎

COROLLARY 4.3 (Filtered-X Algorithm) ***Assume that the adaptive system with har-*** monic *regressor (2. 1)-(2. 5) is* specified *as the Filtered-X algorithm (cf., [25]),* using *gradient adaptation,* **i.e.,**

$$\dot{w} = \mu \tilde{x}(t)e(t) \tag{4.5}$$

$$\tilde{x} = F(p)x(t) \tag{4.6}$$

for some choice *of regressor filter F(s).*

Then, if the XO condition *of Theorem S.1 is satisfied, the LTI expression (3.18) for* $\overline{H}$ *is* given *by,*

$$\overline{H}(s) = \hat{H}(s) \triangleq \mu \sum_{i=1}^{m} d_i{}^2 \cdot \frac{F_R(i)s + F_I(i)\omega_i}{s^2 + \omega_i^2} \tag{4.7}$$

Proof: Result (4.7) follows by substituting, $\Gamma(s) = \frac{1}{s}$ in Theorem 3.1, and rearranging. ∎

COROLLARY 4.4 (Augmented Error (AE) Algorithm) *Assume that the adaptive system with harmonic regressor (1?. 1)-(2.5) is* specified *as the Augmented Error algorithm (cf., [17], [21]), using the gradient adaptation algorithm,* **i.e.,**

$$w = \mu \tilde{x}(t)\epsilon(t) \tag{4.8}$$

where the augmented error ϵ is given *by,*

$$\epsilon = e + F(p)[\hat{y}] - \tilde{y} \tag{4.9}$$

$$\hat{y} = w^T x \tag{4.10}$$

$$\tilde{y} = w^T \tilde{x} \tag{4.11}$$

$$\tilde{x} = F(p)[x] \tag{4.12}$$

for some choice of regressor jilter F(p).

Then, if the XO condition of Theorem S. 1 is satisfied, the mapping from e to $\hat{y}$ is LTI and is given *by,*

$$\overline{H}(s) = \hat{H}(s)(1 + \hat{C}(s) - F(s)\hat{H}(s))^{-1} \tag{4.13}$$

where $\hat{H}(s)$ is defined in *(4. 7),* and,

$$\hat{C}(s) = \mu \sum_{i=1}^{m} d_i^2 \cdot \frac{|F(j\omega_i)|^2 s}{s^2 + \omega_i^2} \tag{4.14}$$

Proof: Using (4.8) and (4.10) together, the mapping from ϵ to $\hat{y}$ can be simply recognized as the Filtered-X algorithm with filter *F(s)* and can be calculated with the aid of Corollary 4.3 to give,

$$\hat{y} = \hat{H}(p)\epsilon \tag{4.15}$$

where $\hat{H}(s)$ is given by (4.7). Similarly, using (4.8) and (4.11) together, the mapping from ϵ to $\tilde{y}$ is of the form of a gradient algorithm with regressor $\tilde{x} = \mathcal{X}\mathcal{F}c(t)$, where $\mathcal{F}$ is defined as,

$$\mathcal{F} \triangleq blockdiag\{\mathcal{F}_i\} \in R^{2m\times 2m} \tag{4.16}$$

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2}; \quad \text{for } i = 1,\ldots,m \tag{4.17}$$

The mapping from ϵ to $\tilde{y}$ can be calculated with the aid of (4.4) in Corollary 4.2 to give,

$$\tilde{y} = \hat{C}(p)\epsilon \tag{4.18}$$

where $\hat{C}(s)$ is given by (4.14), since the related XO condition is satisfied with,

$$\mathcal{F}^T\mathcal{X}^T\mathcal{X}\mathcal{F} = blockdiag\{d_i^2 \cdot |F(j\omega_i)|^2 I_{2\times 2}\} \tag{4.19}$$

Substituting (4.15) and (4. 18) into (4.8) gives upon rearranging,

$$\epsilon = \left(1 + C(p) - F(p)\hat{H}(p)\right)^{-1} e \tag{4.20}$$

Substituting (4.20) into (4.15) gives the desired result (4.13). ∎

5 THE LTI/LTV DECOMPOSITION

The next result shows that in the general case where the XO condition is not satisfied, the mapping $\mathcal{H}$ can always be decomposed into a *parallel connection* of an LTI subsystem and a norm-bounded LTV perturbation.

THEOREM 5.1 (LTI/LTV Decomposition) *Consider the adaptive system (2.1)-(2.3) with harmonic regressor (.%4)($.$). Then,*

(i) *In general the mapping $\mathcal{H}$ from e to $\hat{y}$ can be expressed as the parallel connection of an LTI block $\overline{H}(s)$, and an LTV perturbation block $\tilde{\Delta}$,*

$$\mathcal{H}: \quad \hat{y} = \overline{H}(p)e + \tilde{\Delta}[e] \tag{5.1}$$

where,

$$\overline{H}(s) \triangleq \mu \sum_{i=1}^{m} d_i^2 \cdot H_i(s) \tag{5.2}$$

$$\tilde{\Delta}[e] \triangleq \mu c(t)^T \Delta \Gamma(p) \Big[\mathcal{F} c(t) e \Big] \tag{5.3}$$

$$\Delta \triangleq \mathcal{X}^T \mathcal{X} - D^2 \tag{5.4}$$

$$\mathcal{F} \triangleq blockdiag\{\mathcal{F}_i\} \in R^{2m \times 2m} \tag{5.5}$$

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2}; \qquad \text{for } i = 1, \ldots m \tag{5.6}$$

$$F_R(i) \triangleq \mathbf{Re}(F(j\omega_i)); \quad F_I(i) \triangleq \mathbf{Im}(F(j\omega_i)) \tag{5.7}$$

and where Hi(s) is as defined in (9.19) of Theorem S. 1, and D is chosen (non-uniquely) as any matrix of the pairwise diagonal form (3. 17).*

(ii) If the adaptation law $\Gamma(s)$ is stable with infinity norm $||\Gamma(s)||_\infty$, then the gain of the LTV perturbation can be bounded from above as,

$$||\tilde{\Delta}||_{2i} \leq \mu m \overline{\sigma}(\Delta) ||\Gamma(s)||_\infty \max_i |F(\omega_i)| \tag{5.8}$$

where $||.||_{2i}$ denotes the induced $\mathcal{L}_2$-norm of the indicated operator.

Proof:
Proof of (i): Substituting (2.4) and the relation $\tilde{x} = F(p)[\mathcal{X}c(t)] = \mathcal{X}\mathcal{F}c(t)$ into *(2.1)-(2,3) gives,*

$$\hat{y} = \mu x(t)^T \Gamma(p) \Big[\tilde{x}(t) e \Big] \tag{5.9}$$

$$=\mu c(t)^T \mathcal{X}^T \mathcal{X} \cdot \ \Gamma(p)\Big[\mathcal{F} c(t) e\Big] \tag{5.10}$$

Decompose $\mathcal{X}^T\mathcal{X}$ into two distinct parts using the identity,

$$\mathcal{X}^T\mathcal{X} = D^2 + (\mathcal{X}^T\mathcal{X} - D^2) = D^2 + \Delta \tag{5.11}$$

Substituting identity (5.11) into (5.10), and expanding gives two distinct subsystems,

$$\hat{y} = \mu c(t)^T D^2 \cdot \Gamma(p)\Big[\mathcal{F} c(t) e\Big] + \mu c(t)^T \Delta \cdot \Gamma(p)\Big[\mathcal{F} c(t) e\Big] \tag{5.12}$$

By the results of Theorem 3.1 the LTI part $\overline{H}(s)$ is uniquely associated with the operator containing the D^2 term, and the LTV part $\tilde{\Delta}$ is uniquely associated with the operator containing the Δ term in (5,12).

Proof of (ii): This result follows by standard signal norm bounding methods, and only a brief outline is given, Let,

$$y_\Delta \triangleq \tilde{\Delta}[e] = \mu c^T \Delta \Gamma(p)[\mathcal{F} c e] = \mu ||\beta^T \eta||_2 \tag{5.13}$$

where,

$$\beta \triangleq \Delta^T c \tag{5.14}$$

$$\eta \triangleq \Gamma(p)\,[\mathcal{F} c e] \tag{5.15}$$

and the 2-norm is defined as $||x||_2 \triangleq \left[\int_0^\infty x^T x\,dt\right]^{\frac{1}{2}}$. Then,

$$||y_\Delta||_2 = \mu ||\beta^T \eta||_2 \leq \mu ||\eta||_2 \max_t (\beta^T \beta)^{\frac{1}{2}} \tag{5.16}$$

But it can be shown that,

$$\max_t (\beta^T \beta)^{\frac{1}{2}} \leq m^{\frac{1}{2}} \cdot \overline{\sigma}(\Delta) \tag{5.17}$$

and,

$$||\eta||_2 \leq ||\Gamma(s)||_\infty m^{\frac{1}{2}} \max_i |F(\omega_i)| \cdot ||e||_2 \tag{5.18}$$

Combining (5.16)(5.17) and (5.18) gives,

$$||y_\Delta||_2 \leq \mu m \overline{\sigma}(\Delta) ||\Gamma(s)||_\infty \max_i |F(\omega_i)| \cdot ||e||_2 \tag{5.19}$$

Hence.

$$||\tilde{\Delta}||_{2i} \triangleq \sup_{e \in \mathcal{L}_2} \frac{||y_\Delta||_2}{||e||_2} \leq \mu m \overline{\sigma}(\Delta) ||\Gamma(s)||_\infty \max_i |F(\omega_i)| \tag{5.20}$$

which is the desired result. ■

The LTI/LTV decomposition of $\mathcal{H}$ can be understood by the sequence of block diagram rearrangements shown in Figure 5.1. Specifically, Figure 5.1 Part a. shows the initial

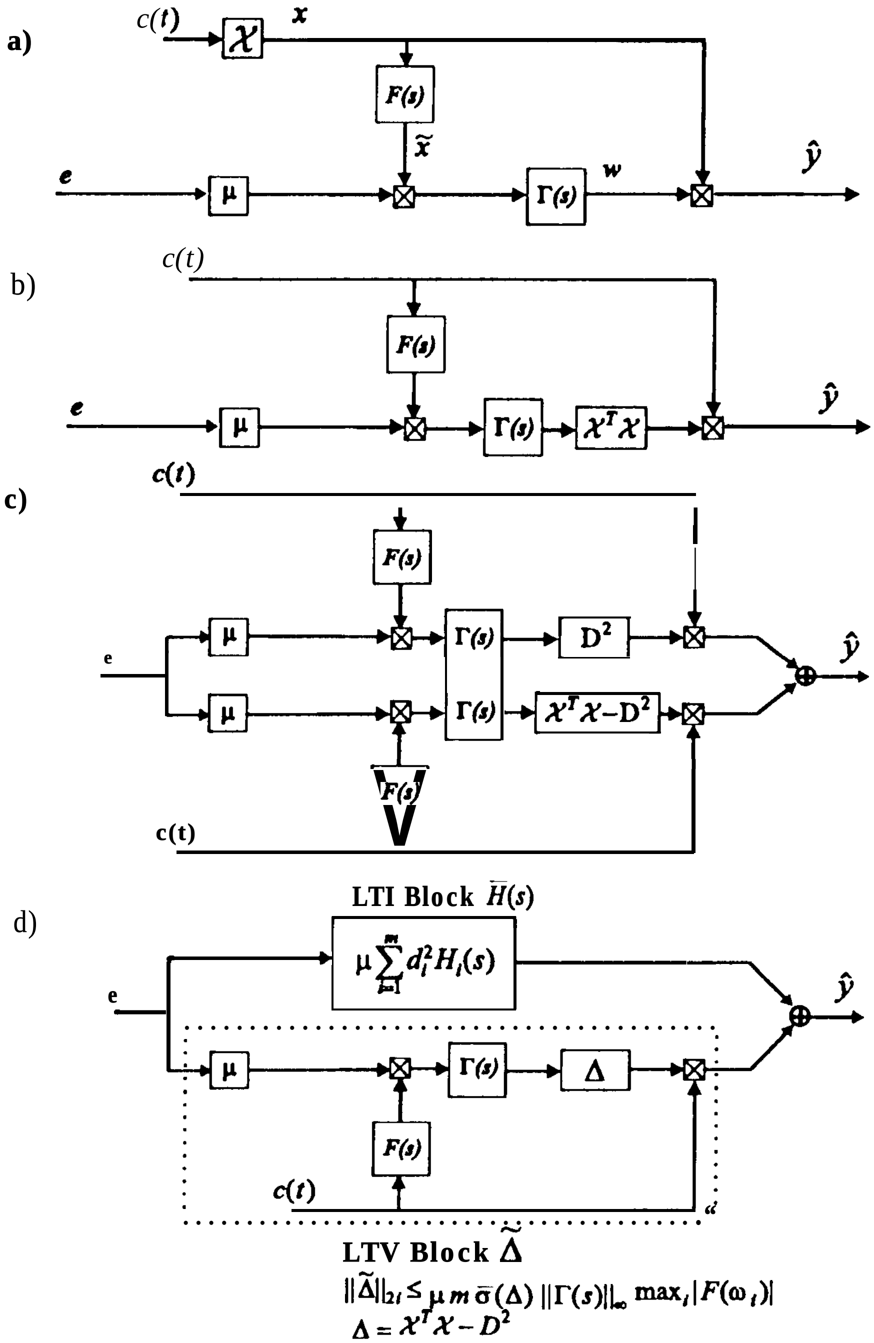


Figure 5.1: General LTI/LTV Decomposition of $\mathcal{H}$

adaptive system with harmonic regressor; Part b. shows the matrix $\mathcal{X}$ pushed through several scalar matrix blocks of the diagram; Part c. uses the identity $\mathcal{X}^T\mathcal{X} = D^2 + (\mathcal{X}^T\mathcal{X} - D^2)$ to split the diagram into two subsystems; and Part d. recognizes the upper subsystem as LTI and the lower subsystem as LTV (from Theorem 3.1) with the indicated norm bound.

REMARK 5.1 The LTI/LTV decomposition in Theorem 5.1 is important for adaptive systems which do not exactly satisfy the XO condition. In this case, the adaptive system can be analyzed using modern robust control methods (i.e., small gain theorem) making use of the analytic expression (5.2) for the LT1 block $\overline{H}(s)$ and the norm bound (5.8) on the time-varying perturbation block $\tilde{\Delta}$ [24] [26]. ■

REMARK 5.2 The need for $\|\Gamma(s)\|_\infty$ to exist in Theorem 5.1 (part ii) requires that the adaptive law use some type of "leakage" (cf., Ioannou and Kokotovic [15]). The possibility of less conservative norm-bound remains as an open issue. ■

6 OPTIMIZED NORM BOUNDS

The decomposition as stated in Theorem 5.1 is only unique for a specified choice of D^2. Hence, D^2 plays the role of a "multiplier" which should be optimized to capture "most" of the LTI character of the $\mathcal{H}$ operator in the LTI/LTV decomposition. The optimization problem will be addressed in this section.

The approach is to minimize the norm-bound (5.8) of the LTV operator over all possible D^2 of the appropriate pairwise diagonal form (3.17). Since the matrix D^2 only appears in the $\bar{\sigma}(\Delta)$ term, this is equivalent to minimizing $\bar{\sigma}(\mathcal{X}^T\mathcal{X} - D^2)$. The problem is stated below and shown to lead to a convex linear matrix inequality (LMI) optimization problem.

LEMMA **6.1** (LTV Norm-Bound Optimization) **Consider** *the following optimization problem,*

$$\min_{\mathcal{D}} \quad \bar{\sigma}(\mathcal{X}^T\mathcal{X} - \mathcal{D}) \tag{6.1}$$

subject to,

$$\mathcal{D} \triangleq D^2 \triangleq \begin{bmatrix} d_1^2 \cdot I_{2\times 2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_m^2 \cdot I_{2\times 2} \end{bmatrix} \in R^{2m\times 2m} \tag{6.2}$$

where, $d_i^2 \geq 0$, $i = 1, \ldots, m$ are arbitrary scalars.

Then the solution is given by solving the following equivalent convex optimization problem,

$$\min_{t,\mathcal{D}} \quad t \tag{6.3}$$

subject to,

$$\begin{bmatrix} t \cdot I & \mathcal{X}^T\mathcal{X} - \mathcal{D} \\ \mathcal{X}^T\mathcal{X} - \mathcal{D} & t \cdot I \end{bmatrix} \geq 0 \tag{6.4}$$

$$\mathcal{D} \geq 0 \tag{6.5}$$

where $\mathcal{D}$ is constrained to have the pairwise diagonal structure (6.2)

Proof: Consider the related optimization problem,

$$\min_t t \tag{6.6}$$

subject to,

$$\begin{bmatrix} t \cdot I & \mathcal{X}^T\mathcal{X} - \mathcal{D} \\ \mathcal{X}^T\mathcal{X} - \mathcal{D} & t \cdot I \end{bmatrix} \geq 0 \tag{6.7}$$

$$t > 0 \tag{6.8}$$

Given $t > 0$, inequality (6.7) is known to be equivalent to $S \geq 0$ where S is the Schur complement $t - (\mathcal{X}^T\mathcal{X} - \mathcal{D})^T(t^{-1} \cdot I)(\mathcal{X}^T\mathcal{X} - \mathcal{D})$ (cf., [9]). But inequality $S \geq 0$ is equivalent to the inequality $t^2 \geq (\mathcal{X}^T\mathcal{X} - \mathcal{D})^T(\mathcal{X}^T\mathcal{X} - \mathcal{D})$, which is minimized by $t = \overline{\sigma}(\mathcal{X}^T\mathcal{X} - \mathcal{D})$. The result of the lemma follows by further optimizing this solution over $\mathcal{D}$ with the constraint $\mathcal{D} \geq 0$. ■

The optimization problem (6.3)-(6.5) is in a standard form of a linear objective function with LMI constraints. As such, it can be solved using many available software packages for LMI problems, such as the LMI Control Toolbox [13].

For single-tone problems, the following result shows that the optimal $D^2 = d^2 \cdot I$ can be found analytically.

LEMMA *6.2* **(Single-Tone Case)** *Consider the optimization problem,*

$$\min_{d^2} \overline{\sigma}(\mathcal{X}^T\mathcal{X} - d^2 \cdot I_{2\times 2}) \tag{6.9}$$

where $d^2 \geq 0$ is an arbitrary scalar.

Then the solution d^2 is given by the average of the diagonals of $\mathcal{X}^T\mathcal{X}$, i.e.,

$$d^2 \triangleq \frac{1}{2}(m^{11} + m^{22}) \tag{6.10}$$

where,

$$\mathcal{X}^T\mathcal{X} \triangleq \mathcal{M} \tag{6.11}$$

$$\mathcal{M} \triangleq \begin{bmatrix} m^{11} & m^{12} \\ m^{21} & m^{22} \end{bmatrix} \in R^{2\times 2} \tag{6.12}$$

Proof: The singular values al, σ_2 of the 2 x 2 symmetric matrix A = $\mathcal{X}^T\mathcal{X}$ – &. *I can* be written in terms of its eigenvalues as,

$$\sigma_i(\Delta) = |\lambda_i(\Delta)| = |\alpha_i(\mathcal{X}^T\mathcal{X}) - d^2|, i = 1,2 \tag{6.13}$$

where $\lambda_i(\Delta)$ denotes an eigenvalue of A and $\alpha_i(\mathcal{X}^T\mathcal{X})$ denotes an eigenvalue of $\mathcal{X}^T\mathcal{X}$. Here, the eigenvalues λ_i and α_i are related by the shift in the complex plane i.e., $\lambda_i = \alpha_i - d^2, i = 1,2$. Hence as d^2 is increased, the λ_i are determined by shifting the (nonegative real) eigenvalues α_i to the left along the real axis a distance of d^2. The quantity $\overline{\sigma}(\Delta) = max(|\lambda_1|,$ [A,1) is clearly minimized at the point where $\lambda_1 = -\lambda_2$, or equivalently where,

$$d^2 = (\alpha_1 + \alpha_2)/2 = Trace(\mathcal{X}^T\mathcal{X})/2 \tag{6.14}$$

which is the desired result (6.10). ∎

7 EXAMPLES

7.1 Imperfect Sin/Cos Regressor

Consider the gradient adaptive algorithm with leakage, i.e.,

$$\dot{w} = -\sigma w + x(t)e(t) \tag{7.1}$$

for some value of the leakage parameter $\sigma \geq 0$. This corresponds to the choice $\Gamma(s) = 1/(s+\sigma), F(s) = 1$ in the adaptive system (2.1)-(2.5).

The ideal sine/cosine regressor is defined by,

$$x = \begin{bmatrix} \sin\omega_1 t \\ \cos\omega_1 t \end{bmatrix}$$

Since $z = \mathcal{X}_1 c(t)$ with $\mathcal{X}_1 = diag[1,1]$ it follows that the XO condition is satisfied exactly with confluence matrix $\mathcal{X}_1^T\mathcal{X}_1 = D^2 = diag[1,1]$. Using the results from Corollary 4.1 the adaptive system is LTI with transfer function,

$$\overline{H}(s) = \mu \frac{s+\sigma}{s^2 + 2\sigma s + (\omega_1^2 + \sigma^2)} \tag{7.2}$$

Now consider the non-ideal regressor case where the amplitude is perturbed by ϵ and the phase is perturbed by θ as follows,

$$x = \begin{bmatrix} (1+\frac{\epsilon}{2})^{\frac{1}{2}} \sin(\omega_1 t + \frac{\theta}{2}) \\ (1-\frac{\epsilon}{2})^{\frac{1}{2}} \cos(\omega_1 t - \frac{\theta}{2}) \end{bmatrix}$$

$$\mathcal{X} = \begin{bmatrix} (1+\frac{\epsilon}{2})^{\frac{1}{2}} & 0 \\ 0 & (1-\frac{\epsilon}{2})^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$$

Then,

$$X^TX = \begin{bmatrix} 1+\frac{\epsilon}{2}\,\mathrm{cm(e)} & \sin(\theta) \\ \sin(0) \quad 1 & -\frac{\epsilon}{2}\ \mathrm{ins(a)} \end{bmatrix} \triangleq D^2 + \Delta$$

where,

$$_{\iota\,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \zeta = \begin{bmatrix} \frac{\epsilon}{2}\cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\frac{\epsilon}{2}\cos(\theta) \end{bmatrix}$$

Here, the choice of D^2 has been optimized using the results of Lemma 6.2 for the single tone case. Clearly, the XO condition is not satisfied since A # O. However, one can compute,

$$\bar{\sigma}(\Delta) = \left(\frac{\epsilon^2}{4}\cos^2(\theta) + \sin^2(\theta)\right)^{\frac{1}{2}}$$

Hence, by the LTI/LTV decomposition the adaptive system is representable by a parallel connection of the LTI block $\overline{H}(s)$ given in (7.2) and an LTV perturbation block $\tilde{\Delta}$ with induced 2-norm bound,

$$\begin{aligned} \|\tilde{\Delta}\|_{2i} &\leq \mu m \bar{\sigma}(\Delta)\|\Gamma(s)\|_\infty |F(j\omega_1)| \\ &= \frac{\mu}{\sigma}\left(\frac{\epsilon^2}{4}\cos^2(\theta) + \sin^2(\theta)\right)^{\frac{1}{2}} \end{aligned}$$

The equivalent LTI system shown in Figure 7.1. It is seen that as ϵ + O and $\theta \to$ O, the norm bound $\tilde{A}$ goes to zero, which ensures a pure LTI representation in the limiting case of a perfect regressor implementation. For the nonideal case of finite ϵ and 6, the above LTI/LTV decomposition is amenable to analysis using standard robust control methods.

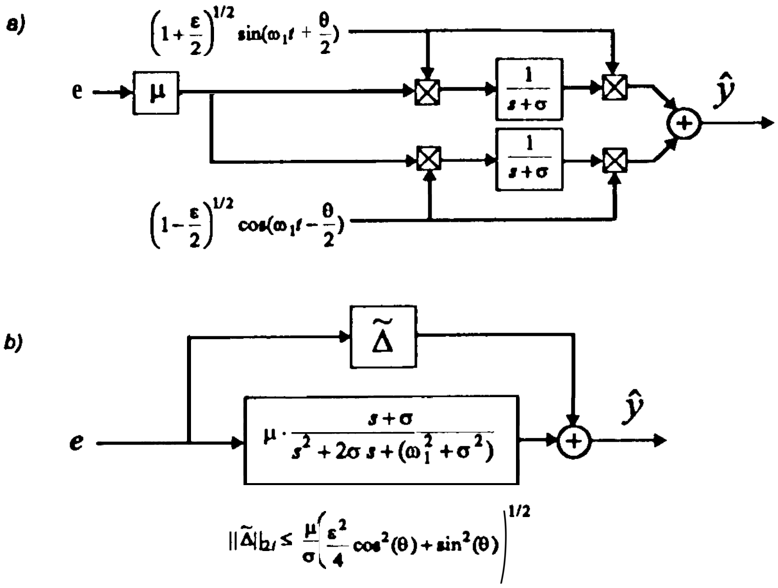


Figure 7.1: LTI/LTV decomposition of an adaptive system with an imperfect sin/cos regressor

7.2 Tap Delay-Line Regressor

Consider the tap delay-line regressor driven by a sum of sinusoids shown in Figure 7.2, Part a.. Here m is the number of tones, T is the tap delay, and $2\underline{\nu}/T$ is the minimal spacing between any two sinusoids $\omega_i \# \omega_j$ in ξ, and the adaptive gain $\mu = \overline{\mu}/N$ has been normalized by the number of taps N.

Applying the LTI/LTV decomposition to this example gives the parallel connection of the LTI block and LTV blocks shown in Figure 7.2 Part b. (see [2] [5] for proof). It is seen that the norm bounded LTV perturbation $\tilde{\Delta}$ can be made arbitrarily small by increasing the number of taps N. Glover's LTI representation [14] is recovered in the limit as the number of taps is increased to infinity.

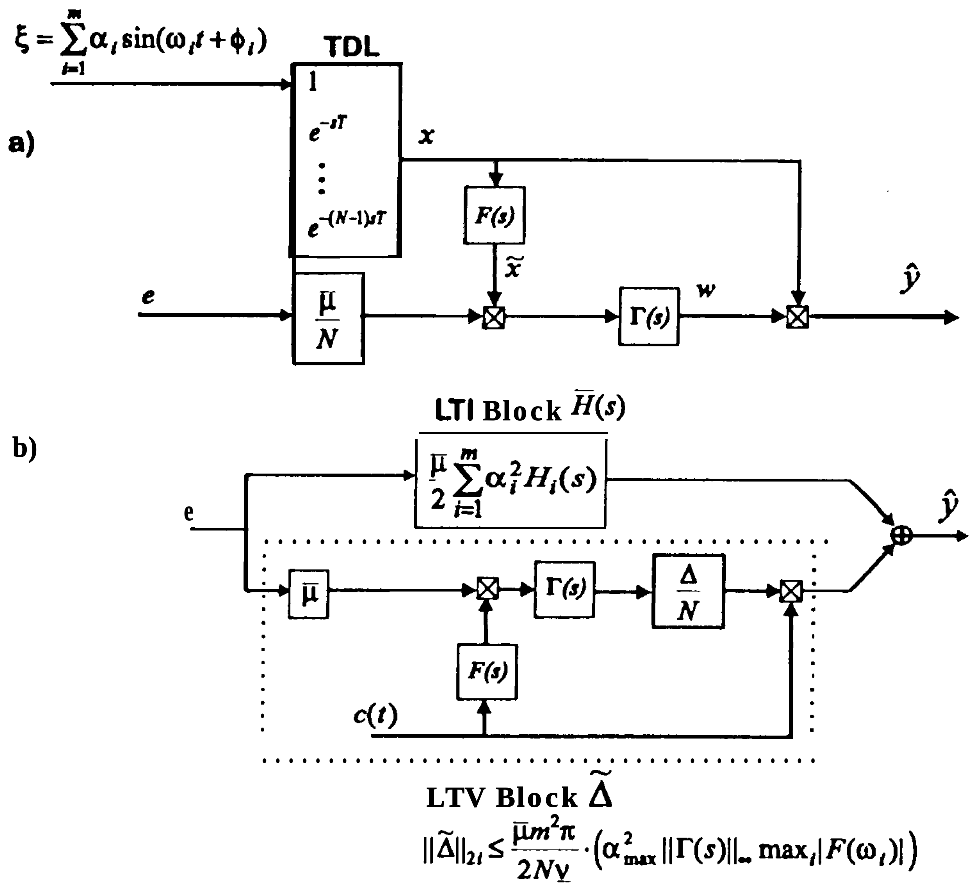


Figure 7.2: LTI/LTV decomposition of $\mathcal{H}$ for harmonic adaptive system with TDL basis

8 DISCUSSION

At this point, several comments are in order.

1. All of the LTI transfer functions $\overline{H}(s)$ in the Corollaries of Section 4 have large gains in the vicinity of the tone frequencies $\omega_i, i = 1,,.., m$. When used in closed-loop, the large gains become "notches" of the form $(1 + \overline{H}(s))^{-1}$. Such closed-loop notches have been shown to be effective at canceling sinusoidal disturbances in a wide variety of adaptive feedforward control applications (cf., Sievers and von Flotow [22], Morgan [19], Collins [10], Spanos and Rahman [23], Bodson, Sacks and Khosla [8], and Messner and Bodson [16]). In light of the new LTI results, many of the conditions and assumptions made in these references can be relaxed.

2. The LTI properties shown for the Augmented Error algorithm in Corollary 4,4 are new, and do not seem to have any counterpart in the existing literature, This is somewhat surprising since the AE algorithm provides an important alternative to the Filtered-X algorithm when there is a plant blocking the noise cancellation path (cf., [2]).

3. The well-known "small μ" constraint imposed on the Filtered-X algorithm for stability (when there is a plant blocking the noise cancellation path), is motivated heuristically by Widrow [25] based on the need to interchange certain LTV blocks in the stabilty proof. However, this constraint can be completely understood and quantitatively determined in an LTI context using Nyquist analysis, Details can be found in [2], Furthermore, an LTI based Nyquist analysis indicates that the small μ constraint is not required for the AE algorithm, and consequently the adaptation gain μ can be made arbitrarily large without causing instability.

4. The fact that a "tall" matrix $\mathcal{X}$ can satisfy the XO condition indicates that even overpararnetrized systems can be LTI and have exponentially convergent tracking errors. Such exponential convergence properties are surprising in light of the fact that the regressor is overparametrized and is not persistently exciting. Interestingly, it is shown in [6] that exponential tracking error convergence is a property of any overparametrized adaptive system with a *positive definite confluence matrix*, and is not fundamentally restricted to systems with LTI representations.

9 CONCLUSIONS

This paper establishes a necessary and sufficient condition for an adaptive system with a regressor composed completely of sinusoids to admit an exact LTI representation. The condition (denoted as the "XO" condition), is simply that the confluence matrix is pairwise diagonal, This condition is equivalent to the property that the block diagram from e to $\hat{y}$ of the adaptive system can be rearranged so that the regressor has a minimal length, persistently exciting, paired sine/cosine regressor. The reduced block diagram is said to be in *Tonal Canonical Form,* The theory reproduces as special cases all known instances of LTI adaptive systems found in the literature, and indicates a much larger class than previously known,

Several LTI related properties were investigated, including (1) invariance of the XO condition under orthogonal regressor transformations; (2) systematic regressor transformations to ensure that the XO condition is satisfied; (3) LTI representations of the Augmented Error algorithm of Monopoli.

The theory was then extended to the applications where the XO condition does not hold. For this case, an LTI/LTV Decomposition Theorem was proved which decomposed the adaptive system into a parallel connection of an LTI subsystem and an LTV subsystem. An explicit norm-bound was established on the LTV subsystem, enabling analysis by robust control methods applicable to LTI systems with norm-bounded perturbations. Since the multiplier matrix D^2 associated with the LTI/LTV decomposition is non-unique, it is best chosen to minimize the size of the norm-bounded LTV perturbation. It was shown that this problem could be formulated as a linear matrix inequality and readily solved using available software. For the single-tone case, an analytic solution was provided for the optimal multiplier.

Two examples were given to demonstrate the new results. The first example used a regressor constructed from an imperfect sine/cosine basis. This example clearly showed how the norm bounded perturbation increases as the sin/cos basis is detuned in amplitude and phase. The second example considered a Tap Delay-Line regressor basis. It waa shown that the adaptive system could be represented as an LTI system with an additive norm-bounded LTV perturbation which decreases as N^{-1} where N is the number of taps. The TDL example puts Glover's 1977 results into a modern control context by providing a valid representation of the adaptive system for any "finite" number of taps, and by exposing the precise nature of convergence to an LTI system as the number of taps becomes large.

The LTI representations developed in this paper are significantly different from other representations commonly used in the adaptive control literature [21]. Specifically, the LTI/LTV representation can be analyzed and designed using modern robust control tools applicable to LTI systems with norm-bounded perturbations. It is hoped that this new representation will lead to a better understanding of how such adaptive systems work, provide an improved characterization of their performance and robustness properties, and lead to new architectures and adaptive design techniques in the future.

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A APPENDIX: Proof of Theorem 3.1

Define,

$$\mathcal{X}^T\mathcal{X} \triangleq \mathrm{M} = \{\mathrm{M}_{ij}\} \in R^{2m\times 2m} \tag{A1}$$

$$\mathrm{M}_{ij} \triangleq \begin{bmatrix} \mathrm{m}_{ij}^{11} & \mathrm{m}_{ij}^{12} \\ \mathrm{m}_{ij}^{21} & \mathrm{m}_{ij}^{22} \end{bmatrix} \in R^{2\times 2}; \qquad \text{for } i,j = 1,...,m \tag{A.2}$$

Using (2.3)-(2.5), the filtered regressor can be represented as,

$$\tilde{x} = F(p)[x] = F(p)[\mathcal{X}c(t)] = \mathcal{X}\mathcal{F}c(t) \tag{A.3}$$

where $\mathcal{F}$ is the block diagonal matrix given by,

$$F = blockdiag\{\mathcal{F}_i\} \in R^{2m\times 2m} \tag{A.4}$$

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2}; \qquad \text{for } i = \mathrm{I},..,)\mathrm{m} \tag{A.5}$$

Proof of (i): It is desired to show that M = D^2 (where D^2 has the block-diagonal form (3.17)), if and only if the mapping $\mathcal{H}$ from e to $\hat{y}$ is LTI. From (2,1)-(2,5) and (A.3) this mapping can be written as,

$$\begin{aligned} \hat{y} &= \mu c(t)^T\mathcal{X}^T \cdot \Gamma(p)[\mathcal{X}\mathcal{F}c(t)e] && \text{(A.6)} \\ &= \mu c(t)^T\mathcal{X}^T\mathcal{X}\mathcal{F} \cdot I'(p)\ [c(t)e] && \text{(A.7)} \\ &= \mu c(t)^T\mathrm{M}\mathcal{F}\int_0^t \gamma(\tau)c(t-\tau)e(t-\tau)d\tau && \text{(A.8)} \\ &= \mu\int_0^t \gamma(\tau)c(t)^T\mathcal{V}c(t-\tau)e(t-\tau)d\tau && \text{(A.9)} \end{aligned}$$

where $\gamma(t)$ is the impulse response of the filter I'(s), and where we have defined the matrix,

$$\mathcal{V} = \mathrm{M}\mathcal{F} \tag{A.1O}$$

For later convenience, V is partitioned into 2 x 2 blocks (compatibly with $\mathcal{F}$,M), as follows,

$$V = \{V_{ij}\} \in R^{2m\times 2m} \tag{A11}$$

$$\mathcal{V}_{ij} \triangleq \begin{bmatrix} v_{ij}^{11} & v_{ij}^{12} \\ v_{ij}^{21} & v_{ij}^{22} \end{bmatrix} \in R^{2\times 2}; \qquad \text{for } i,j = 1,...,m \tag{A.12}$$

It is seen that the mapping $\mathcal{H}$ from e to $\hat{y}$ in (A.9) is represented by a convolution integral, which is time-invariant if and only if the kernel is independent of time t, equivalently, if and only if,

$$c(t)^T\mathcal{V}c(t-\tau) = \beta(\tau) \tag{A.13}$$

where $\beta(\tau)$ is a *function purely of* τ. Condition (A. 13) will be examined in detail. Expanding $c(t-\tau)$ gives the identity,

$$c(t-\tau) = Q(t)c(\tau) \tag{A.14}$$

where $Q(t)$ is the block diagonal matrix,

$$Q(t) = blockdiag\{Q_i(t)\} \in R^{2m\times 2m} \tag{A.15}$$

$$Q_i(t) \triangleq \begin{bmatrix} -\cos\omega_i t & \sin\omega_i t \\ \sin\omega_i t & \cos\omega_i t \end{bmatrix} \in R^{2\times 2}; \qquad \text{for } i = 1,\dots,m \tag{A.16}$$

Substituting (A.14) into (A.13) gives,

$$\alpha^T(t)c(\tau) = \beta(\tau) \tag{A.17}$$

where,

$$\alpha^T(t) \triangleq c(t)^T \mathcal{V} Q(t) \tag{A.18}$$

Equation (A.17) holds if and only if α is a constant vector, i.e., $\alpha(t) = \alpha^\circ$. To see this, multiply both sides of (A. 17) on the right by $c^T(\tau)$ and integrate with respect to τ over any interval $[\tau_1, \tau_2]$ such that $\int c(\tau)c(\tau)^T d\tau$ is invertible. Such an interval always exists since the components of $c(\tau)$ are linearly independent functions (i.e., sines and cosines of distinct frequencies). The resulting equation can be solved for α, implying that any valid solution α to equation (A.17) must be a constant vector.

Assuming that α is constant, consider relation (A.18) taken two components at a time, i.e.,

$$c_i(t)^T \mathcal{V}_{ij} Q_j(t) = [\alpha_1^\circ, \alpha_2^\circ] \tag{A.19}$$

where $\alpha_1^\circ, \alpha_2^\circ$ are constants and,

$$c_i(t) = [\sin\omega_i t, \cos\omega_i t]^T \tag{A.20}$$

Expanding the first component of (A.19) gives,

$$\begin{aligned}
\alpha_1^\circ &= -\cos(\omega_j t)\sin(\omega_i t)v_{ij}^{11} - \cos(\omega_j t)\cos(\omega_i t)v_{ij}^{21} \\
&+ \sin(\omega_j t)\sin(\omega_i t)v_{ij}^{12} + \sin(\omega_j t)\cos(\omega_i t)v_{ij}^{22} &\text{(A.21)} \\
&= \frac{1}{2}(-\sin(\omega_j + \omega_i)t + \sin(\omega_j - \omega_i)t)v_{ij}^{11} - \frac{1}{2}(\cos(\omega_j - \omega_i)t + \cos(\omega_j + \omega_i)t)v_{ij}^{21} \\
&+ \frac{1}{2}(\cos(\omega_j - \omega_i)t - \cos(\omega_j + \omega_i)t)v_{ij}^{12} + \frac{1}{2}(\sin(\omega_j + \omega_i)t + \sin(\omega_j - \omega_i)t)v_{ij}^{22} &\text{(A.22)} \\
&= \frac{1}{2}\left[(v_{ij}^{22} - v_{ij}^{11})^2 + (v_{ij}^{21} + v_{ij}^{12})^2\right]^{\frac{1}{2}} \sin((\omega_j + \omega_i)t + \phi_{ij}) \\
&+ \frac{1}{2}\left[(v_{ij}^{11} + v_{ij}^{22})^2 + (v_{ij}^{12} - v_{ij}^{21})^2\right]^{\frac{1}{2}} \sin((\omega_j - \omega_i)t + \psi_{ij}) &\text{(A.23)}
\end{aligned}$$

Here, (A.22) follows by expanding (A.21) in terms of sum/difference frequencies; and (A.23) follows by rearrangement. The constant phases ϕ_{ij}, ψ_{ij} can also be calculated, but will not be needed. A similar expression to (A.23) can be calculated by using the second term α_2° in (A.19), but this can be shown to be redundant with (A.23) and will not impose additional constraints.

Case 1: $i \neq j$

First consider the case where $i \neq j$ so that ω_i and ω_j are distinct nonzero frequencies. Then (A.23) is the sum of two sinusoids of distinct frequencies, which is equal to a constant if and only if both terms vanish identically, i.e.,

$$v_{ij}^{11} = v_{ij}^{22}; \quad v_{ij}^{21} = -v_{ij}^{12} \tag{A.24}$$

$$v_{ij}^{11} = -v_{ij}^{22}; \quad v_{ij}^{12} = v_{ij}^{21} \tag{A.25}$$

Equivalently, $v_{ij}^{11} = v_{ij}^{22} = v_{ij}^{21} = v_{ij}^{12} = 0$, which gives,

$$\mathcal{V}_{ij} = 0 \qquad \text{for } i \neq j \tag{A.26}$$

However, from (A.10) and the 2 x 2 partitioned structure of matrices M and $\mathcal{F}$,

$$\mathcal{V}_{ij} = \mathrm{M}_{ij}\mathcal{F}_j \tag{A.27}$$

where $\mathcal{F}_j$ in (A.5) is invertible (since its determinant $|F(j\omega_j)|^2$ is nonzero by assumption). Combining (A.26) and (A.27), and using the invertibility of $\mathcal{F}_j$ gives,

$$\mathrm{M}_{ij} = 0; \qquad \text{for } i \neq j \tag{A.28}$$

Case 2: $i = j$

Next consider the case where $i = j$. Then, equation (A.23) becomes,

$$\begin{aligned} \alpha_1^\circ \;=\; & \frac{1}{2}\left[(v_{ii}^{22} - v_{ii}^{11})^2 + (v_{ii}^{21} + v_{ii}^{12})^2\right]^{\frac{1}{2}} \sin(2\omega_i t + \phi_{ii}) \\ & + \frac{1}{2}\left[(v_{ii}^{11} + v_{ii}^{22})^2 + (v_{ii}^{12} - v_{ii}^{21})^2\right]^{\frac{1}{2}} \sin(\psi_{ii}) \end{aligned} \tag{A.29}$$

The second term of (A.29) is constant, as desired. The first term of (A.29) is sinusoidal of nonzero frequency, which is constant-valued if and only if it vanishes identically, i.e.,

$$v_{ii}^{11} = v_{ii}^{22}; \qquad v_{ii}^{21} = -v_{ii}^{12} \tag{A.30}$$

However,

$$\mathrm{M}_{ii}\mathcal{F}_i = \mathcal{V}_{ii} \tag{A.31}$$

or equivalently (by the invertibility of $\mathcal{F}_i$),

$$M_{ii} = \mathcal{V}_{ii}\mathcal{F}_i^{-1} \tag{A.32}$$

By the symmetry and nonegativity of $M = \mathcal{X}^T\mathcal{X}$ one has,

$$m_{ii}^{21} = m_{ii}^{12} \tag{A.33}$$

$$m_{ii}^{11} \geq 0; \quad m_{ii}^{22} \geq 0 \tag{A.34}$$

Expanding (A.32) using properties (A.30)(A.33) and an analytic expression for $\mathcal{F}_i^{-1}$ gives,

$$\begin{bmatrix} m_{ii}^{11} & m_{ii}^{12} \\ m_{ii}^{12} & m_{ii}^{22} \end{bmatrix} = \mathcal{V}_{ii}\mathcal{F}_i^{-1} = \begin{bmatrix} d_i^{\,2} & \delta_i \\ -\delta_i & d_i^{\,2} \end{bmatrix} \tag{A.35}$$

where,

$$d_i^{\,2} \triangleq (v_{ii}^{11}F_R(i) + v_{ii}^{12}F_I(i))/|F(j\omega_i)|^2 \tag{A.36}$$

$$\delta_i \triangleq (-v_{ii}^{11}F_I(i) + v_{ii}^{12}F_R(i))/|F(j\omega_i)|^2 \tag{A.37}$$

By (A.33) and the special form of the right-hand side of (A.35), it follows that, $m_{ii}^{11} = m_{ii}^{22} \triangleq d_i^{\,2} \geq 0$ and $m_{ij}^{21} = m_{ij}^{12} = 0$, which gives,

$$M_{ii} = \begin{bmatrix} d_i^{\,2} & 0 \\ 0 & d_i^{\,2} \end{bmatrix} \geq 0 \tag{A.38}$$

In summary, the kernel of the convolution (A.9) is a function purely of τ if and only if the i, jth block M_{ij} of the matrix M has the form (A.28) for $i \neq j$, and the form (A.38) for $i = j$. Equivalently, the linear operator $\mathcal{H}$ from e to $\hat{y}$ is time-invariant if and only if M has the block-diagonal form of D^2 in (3,17) of Theorem 3.1, which is the desired result.

Proof of (ii): Substituting $\mathcal{X}^T\mathcal{X} \triangleq M = D^2$ into (A.7) gives,

$$\hat{y} = \mu c(t)^T D^2 \cdot \Gamma(p)\,[\mathcal{F}c(t)e] \tag{A.39}$$

$$= \mu \sum_{i=1}^{m} d_i^{\,2} c_i(t)^T \cdot \Gamma(p)\,[\mathcal{F}_i c_i(t)e] \tag{A.40}$$

$$= \mu \sum_{i=1}^{m} d_i^{\,2} \cdot H_i(p)e \tag{A.41}$$

Here, (A.40) follows by the partitioned structure of $D^2, \mathcal{F}, c(t)$; and (A.41) follows by applying Lemma 3.1 (e.g., compare to (3.9)), separately for each term in the sum (A.40). ■

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